

# DIFFERENTIAL GRADED CATEGORIES AND DELIGNE CONJECTURE

BORIS SHOIKHET

**ABSTRACT.** We formulate and prove a version of Deligne conjecture for any  $n$ -fold monoidal abelian category  $A$  over a field  $k$  of characteristic 0, assuming  $A$  is essentially small, and assuming some compatibility and non-degeneracy conditions are satisfied. The output in our results, which conventionally is a homotopy  $(n+1)$ -algebra, is a Leinster  $(n+1)$ -algebra over  $k$  (defined in Section 2). The proof does not use any transcendental methods, although the passage from the Leinster  $n$ -algebras to the homotopy  $n$ -algebras may require some ones.

Generally speaking, we divide the problem into two parts. The first part, completely solved here, is a construction of a Leinster  $(n+1)$ -algebra over  $k$ , out of an  $n$ -fold monoidal  $k$ -linear abelian category (with some compatibility and non-degeneracy condition). This first part works over  $\mathbb{Q}$  and does not use any transcendental methods. The second part, which we do not consider here, should establish an equivalence between the homotopy categories of Leinster  $n$ -algebras over  $k$  and of homotopy  $n$ -algebras over  $k$ , for a field  $k$  of characteristic 0. The second part does not deal at all with  $n$ -fold monoidal categories; its solution may require usage of some transcendental methods.

As an application, we prove in Theorem 6.1, that the Gerstenhaber-Schack complex of a Hopf algebra over a field  $k$  of characteristic 0, admits a structure of a Leinster 3-algebra over  $k$ .

## 1 INTRODUCTION

### 1.1 DELIGNE CONJECTURE

The statement called in nowadays “the classical Deligne conjecture” was suggested by Pierre Deligne in his 1993 letters to several mathematicians, and currently there are several proofs of it, e.g. [MS], [T2], [KS]. It claims the following.

For an associative algebra  $A$  over a field  $k$  of characteristic 0, the cohomological Hochschild complex  $\mathrm{Hoch}^\bullet(A)$  admits a structure of an algebra over the chain operad  $C_\bullet(E_2, k)$  of the topological little discs operad, such that the induced structure over the homology operad  $H_\bullet(E_2, k)$  on the Hochschild cohomology  $HH^\bullet(A)$  is the one discovered by Gerstenhaber in [G]. (1.1)

Some remarks may be useful. The Hochschild cohomology of an associative algebra  $A$  are defined intrinsically as  $\mathrm{Ext}_{\mathrm{Bimod}(A)}^\bullet(A, A)$ , where  $\mathrm{Bimod}(A)$  stands for the category of  $A$ -bimodules.

Murray Gerstenhaber considered a cup-product  $-\cup-$ , and a Lie bracket  $[-,-]$  of degree -1 (called the Gerstenhaber bracket) on the Hochschild complex of  $A$  and proved, that *on the level of cohomology* they obey the Leibniz rule

$$\{a, b \wedge c\} = \{a, b\} \wedge c \pm b \wedge \{a, c\} \quad (1.2)$$

(for homogeneous  $a, b, c$ ), where  $-\wedge-$  and  $\{-,-\}$  are the operations induced on cohomology by  $-\cup-$  and  $[-,-]$ , correspondingly. This data is called a Gerstenhaber algebra over  $k$ . The operad of Gerstenhaber algebras is an operad in  $k$ -vector spaces, denoted by  $e_2$ .

In 1976, Fred Cohen [C] proved that the operad  $e_2$  is the homology operad of the little discs operad  $E_2$ , for the case of char  $k = 0$ :  $e_2 = H_*(E_2, k)$ .

The situation was as follows: on the cohomology of the Hochschild complex there acts the cohomology operad of the chain operad of little discs. Then Deligne claimed that the chain operad itself (which is an operad in complexes of vector spaces over  $k$ ) acts on the Hochschild complex.

This claim is indeed non-trivial, as the equation analogous to (1.2) fails on the level of Hochschild cochains:

$$[\Psi_1, \Psi_2 \cup \Psi_3] \neq [\Psi_1, \Psi_2] \cup \Psi_3 \pm \Psi_2 \cup [\Psi_1, \Psi_3] \quad (1.3)$$

(for homogeneous  $\Psi_1, \Psi_2, \Psi_3$ ).

A proof of the claim (1.1) was suggested in the Getzler-Jones' 1994 preprint [GJ], but later a mistake in their proof was found.

A new interest to a proof of Deligne conjecture arose after Tamarkin's proof [T1] in 1998 of the Kontsevich formality theorem [Ko] (published one year after the Kontsevich's proof). In Tamarkin's proof, the Deligne conjecture plays a crucial role, and it is the only transcendental step in his proof. Since that, many proofs of it appeared, see [MS], [T2], [KS].

Moreover, it follows from the above proofs, that the chain operad of little discs  $C_*(E_2, k)$  is quasi-equivalent to the operad  $G_\infty$  of homotopy Gerstenhaber algebras.

Thus, in nowadays, the Deligne's formulation (1.1) is commonly replaced by the following one:

**THEOREM 1.1.** *Let  $A$  be an associative algebra (resp., a dg algebra, a dg category) over a field  $k$  of characteristic 0. Then the graded vector space  $\mathrm{RHom}_{\mathrm{Bimod}(A)}^\bullet(A, A)$  admits a structure of algebra over the homotopy Gerstenhaber operad  $G_\infty$ , such that the induced structure over the homology operad  $e_2$  on the Hochschild cohomology  $\mathrm{Ext}_{\mathrm{Bimod}(A)}^\bullet(A, A)$  is the one discovered by Gerstenhaber in '60s.*

## 1.2

Now we warn the reader that, although we prove here a Deligne conjecture in much more general set-up, our results do not imply Theorem 1.1. The matter is that the “output” in our results are

not homotopy  $n$ -algebras, but some algebraic structures called here *Leinster  $n$ -algebras*. They are a particular case of the concept of *Leinster  $n$ -monoids*, introduced by Tom Leinster in [Le]. We will give a detailed definition of them in Section 2.

Morally, Leinster monoids are the cousins of weak Segal monoids [Se], in the case of categories enriched over an arbitrary, non necessarily a cartesian-monoidal, symmetric monoidal category. Let  $\mathcal{M}$  be a set-enriched monoidal category; Graeme Segal introduced *weak monoids in  $\mathcal{M}$* . Here the category of sets can be replaced by any cartesian-monoidal category. However, if we would like to give an analogous definition in the category of  $k$ -vector spaces (of complexes of  $k$ -vector spaces, of differential graded  $k$ -algebras,...) we will experience a trouble. Namely, a genuine monoid in  $\mathcal{V}ect(k)$  does not define a weak Segal monoid. It happens as for two vector spaces  $V, W$ , there no projections  $V \otimes_k W \rightarrow V$ ,  $V \otimes_k W \rightarrow W$ .

An appropriate definition given in [Le] seems to be nice, but currently the author does not know how to prove that the homotopy categories of Leinster  $n$ -algebras over  $k$ ,  $\text{char } k = 0$ , and of homotopy  $n$ -algebras, are equivalent. This equivalence may require some transcendental methods. We hope to discuss this question in our sequel papers.

After saying above what is the output for our result, we discuss now what is an input.

We prove here a version of Deligne conjecture for arbitrary *monoidal abelian category*, with weak compatibility of the exact and monoidal structures, see Definition 4.1. Moreover, we prove it also for an abelian  $n$ -fold monoidal category, in sense of [BFSV].

Our main result is:

**THEOREM 1.2.** *Let  $\mathcal{A}$  be an essentially small  $k$ -linear abelian  $n$ -fold monoidal category, where  $\text{char } k = 0$ . Let  $e$  be the unit object of  $\mathcal{A}$ . Suppose the weak compatibility of the exact and the monoidal structures, as in Definition 5.1. Suppose, as well, that the  $n$ -fold monoidal structure is non-degenerate, in the sense of Definition 5.2. Then  $\text{RHom}_{\mathcal{A}}^{\bullet}(e, e)$  is a Leinster  $(n+1)$ -algebra, whose underlying Leinster 1-algebra is defined from the Yoneda product in  $\text{RHom}_{\mathcal{A}}^{\bullet}(e, e)$ .*

It is proven in Theorem 4.4 for the case  $n = 1$ , and in Theorem 5.3 for general  $n$ .

Notice that the non-degeneracy condition of Definition 5.2 is empty for  $n = 1$ .

The assumption that  $\mathcal{A}$  is essentially small is certainly too strong. We will replace it to a suitable weaker one in our later papers. We impose it here trying to make the paper more readable, and to skip as much as possible the set-theoretical problems.

The link between this more general input, and the classical one, is as follows: the category of  $A$ -bimodules  $\mathcal{B}imod(A)$  is a  $k$ -linear abelian monoidal category, with the monoidal product  $M \otimes_A N$ ,  $M, N \in \mathcal{B}imod(A)$ , whose two-sided unit is the tautological  $A$ -bimodule  $A$ .

*An important feature of our proof of Theorem 1.2 that it does not use any transcendental methods, and it works well over the field  $\mathbb{Q}$ .* All transcendental constructions may appear when one wants to pass from the Leinster  $n$ -algebras to the homotopy  $n$ -algebras, but Theorem 1.2, as it is formulated, does not use any transcendental methods. It makes possible further applications to a proof of Kontsevich formality over  $\mathbb{Q}$ . We consider it in our next papers.

### 1.3

A starting point for this paper was the author’s reading of the Kock-Toën’s paper [KT].

The loc.cit. proves, in a very conceptual way, some kind of Deligne conjecture for  $n$ -fold monoidal categories *over simplicial sets*. The authors do not prove in loc.cit. the corresponding statement in the  $k$ -linear setting. They experienced the trouble with definitions of Segal monoids. We quote [KT, page 2]:

...It is fair to point out that our viewpoint and proof do not seem to work for the original Deligne conjecture, since currently the theory of Segal categories does not work well in linear contexts (like chain complexes), but only in cartesian monoidal contexts. ... However, our original motivation was not to give an additional proof of Deligne’s conjecture, but rather to try to understand it from a more conceptual point of view.

We make a comment on it. The main trouble in the linear context is that the nerve of a monoid in the category  $\mathcal{Vect}(k)$  is *not* a simplicial set. Here we mean the following. Let  $M$  be a monoid in  $\mathcal{Vect}(k)$ . It is natural to define its nerve using the tensor product  $\otimes_k$  instead of the direct product in the set-enriched case. We define

$$X_n = X \otimes_k X \otimes_k \cdots \otimes_k X \quad (n \text{ factors}) \quad (1.4)$$

Then  $X_\bullet$  is not a simplicial set. The two extreme face maps are ill-defined, as there are no projections  $X^{\otimes n} \rightarrow X^{\otimes(n-1)}$  along the first (corresp., the last)  $n-1$  factors. It can be rephrased saying that the symmetric monoidal category  $\mathcal{Vect}(k)$  is not cartesian-monoidal.

Then the author is realized that the Leinster monoids [Le] provide an appropriate concept for transforming the results of [KT] to the  $k$ -linear set-up.

The Dwyer-Kan localization, used in loc.cit., is replaced here by (a modified version of) Drinfeld dg quotient [Dr2], and the hard result [DK, Corollary 4.7] used in [KT], is replaced by the Drinfeld’s theorem [Dr2] saying that the Keller’s and the Drinfeld’s dg quotient give isomorphic objects in the homotopy category of dg categories.

Here the Keller’s dg quotient [Ke2], [Ke3] is used for a proof that for a pre-triangulated dg category it has the “right homotopy type”, which seems impossible to prove directly with the Drinfeld’s construction of dg quotient, and the Drinfeld’s construction is used for its nice monoidal properties (which are achieved after a mild modification of it, see Section 3.4), which seems impossible to prove directly with the Keller’s construction. Then the mentioned above Drinfeld result guarantees that we have at once the both benefits. Drinfeld proved this result formulating a universal property of dg quotient, and proving that the both constructions fulfil this property.

### 1.4 RELATIONSHIP WITH J.LURIE’S APPROACH

One needs to comment on a relationship of our work with J.Lurie’s DAGVI paper [L1], Section 6.1.4 (see also [L2]).

Lurie works in the setting of  $\infty$ -categories, and he defines the concept of “a  $\mathbb{E}[n]$ -algebra” in the category of  $\infty$ -categories, which is what is corresponded to the  $n$ -fold monoidal categories we work with here. For such  $\mathbb{E}[n]$ -algebras he proves a statement close to the generalized Deligne conjecture.

However, it seems to the author that to associate a  $\mathbb{E}[n]$ -algebra to a particular  $n$ -fold monoidal abelian category in the sense of [BFSV], or, more generally, to a particular “elementary” algebraic question, may be a very hard problem. (Here by “elementary” we mean a question which does not belong a priori to the world of  $\infty$ -categories).

For the simplest case  $n = 1$ , one should ask ourselves when a  $k$ -linear abelian monoidal category produces a 1-monoidal  $\infty$ -category, satisfying the Lurie’s definition. An assumption related with our Definition 4.1 (or with Hovey monoid axioms, etc.) will unavoidably appear at this place. The Deligne conjecture is not true for an arbitrary  $k$ -linear abelian monoidal category.

For the case  $n = 2$ , the above problem might be even harder. In the same time, the  $n$ -fold monoidal abelian categories in sense of [BFSV] arise naturally in many algebraic questions. For instance, we proved in [Sh1], that the abelian category of tetramodules over a bialgebra  $B$  is a 2-fold monoidal category. We prove in Theorem 6.1 of present paper, that when  $B$  is a Hopf algebra, the monoidal structures and the exact structure are weak compatible in the sense of Definition 5.1, and the assumption of Definition 5.2 is also fulfilled. Thus, Theorem 5.3 implies that the deformation complex of a Hopf algebra  $B$  (called the Gerstenhaber-Schack complex of  $B$ ) is a Leinster 3-algebra. We have doubts whether the latter claim is true for an arbitrary bialgebra  $B$  (which is not necessarily a Hopf algebra).

## 1.5 ORGANIZATION OF THE PAPER

Section 1 is the Introduction.

In Section 2 we introduce, following Leinster [Le], the Leinster weak monoids in a symmetric monoidal category. They substitute the Segal weak monoids [Se] for the case when  $\mathcal{M}$  is not necessary cartesian-enriched category. In particular, this concept is well-defined in the linear context.

In Section 3 we review the Keller’s and the Drinfeld’s constructions of dg quotient. The contents of Sections 3.3 and 3.4, where we introduce a minor modification of the Drinfeld’s dg quotient construction to achieve nice monoidal properties, may be new.

We prove Deligne conjecture (in that form how we understand it here) for  $n$ -fold monoidal essentially small abelian categories in Section 4, for  $n = 1$ , and in Section 5 for general  $n$ . The main results are Theorem 4.4 and Theorem 5.3.

In Section 6, we consider an application of our results to deformation theory of associative bialgebras. We prove in Theorem 6.1 that, when a bialgebra  $B$  is a Hopf algebra (that is, it has an antipode), its Gerstenhaber-Schack complex is a Leinster 3-algebra. This result is a further development of our construction of a 2-fold monoidal structure on the category of tetramodules

over a bialgebra  $B$ , established in [Sh1].

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## 2 SEGAL MONOIDS AND LEINSTER MONOIDS

Here we give the definitions of *weak Leinster monoids*, and of *weak Leinster  $n$ -monoids*. We define, as a particular case, the concept of a *Leinster  $n$ -algebra* over a field  $k$ .

There are many models for the concept of a homotopy  $n$ -algebra. For instance, when  $n = 2$ , they include the concepts of dg Gerstenhaber algebra, of a homotopy Gerstenhaber algebra, defined using the Koszul resolution of the Gerstenhaber operad, and of a Getzler-Jones  $B_\infty$  algebra.

They are all “homotopy equivalent” over a field of characteristic 0, but the equivalences themselves is a complicated question. Thus, the passage from  $B_\infty$  algebras to homotopy Gerstenhaber algebras depends on the choice of a Drinfeld associator, as was shown by Tamarkin in [T]. The fact that this passage can be performed over  $\mathbb{Q}$  relies on a very non-trivial result of Drinfeld on the existence of associators over  $\mathbb{Q}$ , proven in [Dr1].

Here we use another model for homotopy  $n$ -algebras, which we call *Leinster  $n$ -algebras*, as it is a particular case of general Segal-Leinster  $n$ -monoids. At the moment, the statement that the model category of Leinster  $n$ -algebras is Quillen equivalent to the model category of homotopy  $n$ -algebras when  $\text{char } k = 0$ , has a status of a conjecture. (In fact, to the best of the author's knowledge, the model structure itself on the category of Leinster  $n$ -algebras is not constructed yet). As well, we conjecture that for  $\text{char } k = p > 0$ , the category of Leinster  $n$ -algebras over  $k$  is Quillen equivalent to the category of  $p$ -Dyer-Lashof  $n$ -algebras.

### 2.1

Recall that a *Segal monoid* in a symmetric monoidal category  $\mathcal{M}$  is a simplicial object  $A: \Delta^{\text{opp}} \rightarrow \mathcal{M}$  such that the natural maps

$$\varphi_n: A_n \rightarrow A_1 \times A_1 \times \cdots \times A_1$$

defined by the iterated extreme boundary maps, are *weak equivalences* (in an appropriate sense), here  $\times$  denotes the monoidal product in  $\mathcal{M}$ .

This definition is well-applicable only when  $\mathcal{M}$  is a cartesian-monoidal category, that is, when  $\mathcal{M}$  admits the cartesian products, and the monoidal and the cartesian products coincide. (For example, the monoidal category  $\mathcal{Vect}(k)$  of vector spaces over a field  $k$ , with  $\otimes_k$  as the monoidal product, is *not* cartesian-monoidal. Indeed, the cartesian product of two vector spaces  $V, W$  is their direct sum  $V \oplus W$ , not the tensor product  $V \otimes_k W$ ).

The reason of the uselessness of this concept for non-cartesian-monoidal categories  $\mathcal{M}$  is that a honest monoid  $A$  does *not* define a Segal monoid, so one can not consider the Segal monoids as a weakening of the honest monoids in non-cartesian case. We firstly recall how a honest monoid defines a Segal monoid in the case when  $\mathcal{M}$  is cartesian-monoidal.

When  $A$  is a honest monoid in a cartesian-monoidal  $\mathcal{M}$ , set

$$A_n = A \times A \times \dots A \quad (n \text{ factors, } n \geq 1), \quad A_0 = *$$

To define a functor

$$A^{\text{Seg}}: \Delta^{\text{opp}} \rightarrow \mathcal{M}, \quad A^{\text{Seg}}([n]) = A_n$$

one needs to define the boundary and the degeneracy maps actions on  $\{A_n\}$ . There is easy to define the degeneracy maps: one substitute iteratively the *unit object*  $e$  in  $A$  to the corresponding places, we do not use that  $\mathcal{M}$  is cartesian-monoidal here. We, however, use it to define the boundary maps. Consider what happens for boundary maps  $A_2 \rightarrow A_1$ .

We have  $A_1 = A$ ,  $A_2 = A \times A$ . There should be *three* boundary maps  $A_2 \rightarrow A_1$ , corresponded to three possible semi-monotonous maps  $[0, 1] \rightarrow [0, 1, 2]$  in  $\Delta$ . Denote these maps by  $\delta_0, \delta_1, \delta_2$ ; that is  $\delta_i$  is the imbedding in whose image  $2 - i$  is omitted,  $i = 0, 1, 2$ .

The corresponding maps  $A_2 \rightarrow A_1$  are defined as

$$\delta_0(a \times b) = a, \quad \delta_1(a \times b) = a * b, \quad \delta_2(a \times b) = b \quad (2.1)$$

where  $*$  is the operation in the monoid  $A$ .

When  $\mathcal{M}$  is not cartesian-monoidal, only the map  $\delta_1$  makes sense, but the maps  $\delta_0, \delta_2$  are ill-defined. Therefore, a honest monoid  $A$  does not define, in the common way, a Segal monoid, when  $\mathcal{M}$  is not cartesian-monoidal.

Tom Leinster suggested in [Le] the following modified definition of a Segal monoid, which fixes this problem. That is, a honest monoid  $A$  in  $\mathcal{M}$  defines a Segal-Leinster monoid  $A^{\text{SL}}$  even when  $\mathcal{M}$  is not cartesian-monoidal. Let us recall the Leinster's definition.

Denote by  $\Delta_f$  (the category of finite intervals) the subcategory of the simplicial category  $\Delta$ , having the same objects  $[0], [1], [2], \dots$  as  $\Delta$ , and the morphisms of  $\Delta_f$  are those semi-monotonous maps  $f: [m] \rightarrow [n]$  for which  $f(0) = 0$  and  $f(m) = n$ . That is the morphisms in  $\Delta_f$  are the morphisms of  $\Delta$  which leave the end-points of the intervals fixed.

The category  $\Delta_f$  (unlike the category  $\Delta$  itself) is *monoidal*. The product is defined on objects as  $[m_1] \otimes [m_2] = [m_1 + m_2]$ , so we glue the right end-point of  $[m_1]$  with the left end-point of  $[m_2]$ . Then it is well-defined on morphisms, due to the fixing of the end-points.

DEFINITION 2.1. (i) Let  $\mathcal{M}$  be a symmetric monoidal category with a class  $T$  of morphisms containing all isomorphisms in  $\mathcal{M}$ . Such  $\mathcal{M}$  is called a *monoidal category with weak equivalences* and the morphisms in  $T$  are called *weak equivalences* if the monoidal product of two morphisms in  $T$  is again a morphism in  $T$ .

(ii) A *Segal-Leinster monoid* in a category  $\mathcal{M}$  with weak equivalences is a colax-monoidal functor  $A: \Delta_f^{\text{opp}} \rightarrow \mathcal{M}$  whose colax maps

$$\beta_{m,n}: A_{m+n} \rightarrow A_m \otimes A_n$$

and

$$\alpha: A([0]) \rightarrow \underline{e}$$

are weak equivalences (here  $e$  is the unit in  $\mathcal{M}$ , and  $\underline{e}$  is a monoid in  $\mathcal{M}$  with single object  $e$ , and with morphisms  $\text{Hom}(e, e)$ ),

(iii) A *Segal-Leinster pre-monoid* in  $\mathcal{M}$  is the same data with skipping the condition that the colax maps are weak equivalences.

DEFINITION 2.2. Let  $A$  be a honest monoid in a monoidal category  $\mathcal{M}$ . The corresponding Segal-Leinster monoid  $A^{\text{SL}}$  is defined as

$$A_n^{\text{SL}} = A \otimes A \otimes \cdots \otimes A \quad (n \text{ factors}) \text{ for } n \geq 1, \quad A_0 = e \quad (2.2)$$

where  $e$  is the unit object in  $\mathcal{M}$ .

Equation (2.2) defines a functor  $A^{\text{SL}}: \Delta_f^{\text{opp}} \rightarrow \mathcal{M}$ . The boundary maps  $\delta_i: A_n^{\text{SL}} \rightarrow A_{n-1}^{\text{SL}}$  are defined as the identity elsewhere the  $i$ th and  $(i+1)$ st factors where it is equal to the product in  $A$ ,  $m: A \otimes A \rightarrow A$ . The degeneracy maps  $\epsilon_i: A_n^{\text{SL}} \rightarrow A_{n+1}^{\text{SL}}$  put the unit  $e$  to the  $i$ th position,  $1 \leq i \leq n+1$ .

This functor has the natural colax-monoidal structure, with

$$\beta_{mn}: A_{m+n}^{\text{SL}} \rightarrow A_m^{\text{SL}} \otimes A_n^{\text{SL}}$$

and

$$\alpha: A_0 \rightarrow e$$

the identity maps.

One immediately sees

LEMMA 2.3. Let  $\mathcal{M}$  be a monoidal category with weak equivalences, and let  $\mathcal{SL}(\mathcal{M})$  be the category of Segal-Leinster monoids in  $\mathcal{M}$ . Define on  $\mathcal{SL}(\mathcal{M})$  the monoidal structure component-wise, using the monoidal structure in  $\mathcal{M}$ . Then  $\mathcal{SL}(\mathcal{M})$  becomes a monoidal category with weak equivalences, whose weak equivalences are the component-wise weak equivalences.



◇

Consider the case  $\mathcal{M} = \mathcal{Vect}^\bullet(k)$ , the category of complexes of  $k$ -vector spaces,  $k$  is a field. The monoidal structure of  $V$  and  $W$  in  $\mathcal{Vect}^\bullet(k)$  is their tensor product  $V \otimes_k W$ .

DEFINITION 2.4. A *Segal-Leinster 1-algebra*  $A$  over field  $k$  is defined as a Segal-Leinster monoid in  $\mathcal{M} = \mathcal{Vect}^\bullet(k)$   $\Delta_f^{\text{opp}} \rightarrow \mathcal{Vect}^\bullet(k)$ ,  $[n] \mapsto A_n$ , with  $A_1 = A$ .

Based on Lemma 2.3, we define the category  $\mathcal{L}(n)$  of Leinster  $n$ -algebras:

DEFINITION 2.5. A Leinster  $n$ -algebra  $A$  is an  $n$ -iterated Segal-Leinster monoid

$$(\Delta_f^{\text{opp}})^{\times n} \rightarrow \mathcal{Vect}^\bullet, \quad [i_1] \times [i_2] \times \cdots \times [i_n] \mapsto A_{i_1 i_2 \dots i_n}$$

which is an iterated colax-monoidal functor with weak equivalence colax-maps, with

$$A_{11\dots 1} = A$$

Here we set  $\mathcal{L}(0) = \mathcal{Vect}^\bullet(k)$ .

### 3 THE KELLER'S AND DRINFELD'S CONSTRUCTIONS OF DG QUOTIENT

#### 3.1 THE DG QUOTIENT OF DG CATEGORIES

The dg quotient  $\mathcal{C}/\mathcal{C}_0$  of a dg category  $\mathcal{C}$  by an *essentially small* full dg subcategory  $\mathcal{C}_0$  was firstly introduced by Bernhard Keller in [K1, K3]. It is a dg category, which in the case when  $\mathcal{C}$  is a pre-triangulated dg category,  $\mathcal{C}_0$  its full essentially small pre-triangulated dg subcategory, has the following description.

PROPOSITION 3.1 (B.Keller, [K3, Section 4]). *Let  $\mathcal{C}$  be a pre-triangulated dg category,  $\mathcal{C}_0$  its essentially small full pre-triangulated dg sub-category. Then is a pre-triangulated dg category  $\mathcal{C}/\mathcal{C}_0$  whose triangulated category*

$$H^0(\mathcal{C}/\mathcal{C}_0) \simeq H^0(\mathcal{C})/H^0(\mathcal{C}_0) \tag{3.1}$$

where the quotient in the right-hand side is the Verdier quotient of triangulated categories.

Morally, the dg quotient is a functor from the category of all pairs of dg categories  $(\mathcal{C}, \mathcal{C}_0)$  with  $\mathcal{C}_0$  essentially small, to dg categories, but having nice homotopy properties only when  $\mathcal{C}$  is pre-triangulated. In the latter case the dg quotient  $\mathcal{C}/\mathcal{C}_0$  has the same image in the homotopy category of dg categories as the Toën dg localization [To1, Section 8.2]  $\mathcal{C}[S^{-1}]$  where  $S$  is the set of closed degree 0 morphisms  $s$  in  $\mathcal{C}$  such that  $\text{Cone}(s) \in \mathcal{C}_0$ .

Vladimir Drinfeld [Dr2] gave another construction of the dg quotient  $\mathcal{C}/\mathcal{C}_0$ , which we recall in Section 3.2. In Drinfeld's construction of dg quotient  $\mathcal{C}/\mathcal{C}_0$ , it has the same objects as  $\mathcal{C}$ .

However, the morphisms from two objects  $X$  and  $Y$  may become complexes non-trivial in many degrees even if they are non-trivial only in degree 0 in  $\mathcal{C}$ .

In our work, we use (a slightly modified version of) the Drinfeld's construction of dg quotient, to establish its *monoidal properties* in Proposition 3.3 below (which seemingly one can not derive from the Keller construction). In the same time, the description of the dg quotient in the pre-triangulated case given in Proposition 3.1 seemingly can be not seen directly from the Drinfeld's construction.

Drinfeld formulated in [Dr2] a universal property, which characterizes a dg quotient uniquely, up to an isomorphism, as an objects of the homotopy category of dg categories  $\mathrm{HoCat}^{\mathrm{dg}}(k)$ . Denote by  $[\mathcal{X}]$  a dg category  $\mathcal{X}$  as the object of the homotopy category  $\mathrm{HoCat}^{\mathrm{dg}}(k)$ . The universal property is the following:

Let  $\mathcal{C} \supset \mathcal{C}_0$  be dg categories,  $\mathcal{C}_0$  essentially small. A morphism  $\overline{F}: [\mathcal{C}/\mathcal{C}_0] \rightarrow [\mathcal{D}]$  in  $\mathrm{HoCat}^{\mathrm{dg}}(k)$  is the same that a morphism  $F: [\mathcal{C}] \rightarrow [\mathcal{D}]$  in  $\mathrm{HoCat}^{\mathrm{dg}}(k)$  such that the corresponding functor  $H^0 F: H^0[\mathcal{C}] \rightarrow H^0[\mathcal{D}]$  of homotopy categories maps the image of  $H^0[\mathcal{C}_0]$  in  $H^0[\mathcal{C}]$  to 0. The latter means, by definition, that  $H^0 F(\mathrm{id}_X)$  for any  $X$  in the image of  $H^0[\mathcal{C}_0]$  in  $H^0[\mathcal{C}]$ , is zero morphism in  $H^0[\mathcal{D}]$ .

Drinfeld proved [Dr2] that both constructions of dg quotient, the Keller's one and the one of himself, obey this universal property, and therefore, define isomorphic objects in  $\mathrm{HoCat}^{\mathrm{dg}}(k)$ . This result is very important for our paper. Roughly speaking, this result plays the same role in our paper, that the hard Dwyer-Kan result [DK, Corollary 4.7] used in the Kock-Toën's non-linear Deligne conjecture [KT].

### 3.2 DRINFELD DG QUOTIENT

Let  $\mathcal{C}$  be a dg category over a field  $k$ , and let  $\mathcal{C}_0$  be its essentially small full dg subcategory. Drinfeld defines [Dr] the dg quotient  $\mathcal{C}/\mathcal{C}_0$  as the following dg category.

The category  $\mathcal{C}/\mathcal{C}_0$  has the same objects as  $\mathcal{C}$ , and the category  $\mathcal{C}$  is imbedded into  $\mathcal{C}/\mathcal{C}_0$  as a dg category. Choose an object  $X$  in  $\mathcal{C}_0$  for any class of isomorphism of objects in  $\mathcal{C}_0$ , these objects  $\{X\}$  define a sub-category  $\overline{\mathcal{C}_0} \subset \mathcal{C}_0$ . The assumption that  $\mathcal{C}_0$  is essentially small guarantees that  $\overline{\mathcal{C}_0}$  is small. For any object  $X$  in  $\overline{\mathcal{C}_0}$ , one introduces a new morphism  $\varepsilon_X$  in  $\mathrm{Hom}(X, X)$  of degree -1, with  $d(\varepsilon_X) = \mathrm{id}_X$ , without any new relations. By definition,  $\mathcal{C}/\mathcal{C}_0$  is the category with objects  $\{\mathrm{Ob}\mathcal{C}\}$ , and with morphisms are obtained by free joint of the algebraic envelope of  $\{\varepsilon_X\}_{X \in \overline{\mathcal{C}_0}}$  to the morphisms of  $\mathcal{C}$ .

One easily sees that, for  $X, Y \in \mathrm{Ob}\mathcal{C}$ , the underlying  $k$ -linear space

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{C}_0}(X, Y) = \oplus \mathrm{Hom}_{\mathcal{C}/\mathcal{C}_0}^{(n)}(X, Y)$$

where

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{C}_0}^{(n)}(X, Y) = \oplus_{Y_0, \dots, Y_{n-1} \in \mathcal{C}_0} \mathrm{Hom}_{\mathcal{C}}(X, Y_0) \otimes k[1] \otimes \mathrm{Hom}_{\mathcal{C}}(Y_0, Y_1) \otimes k[1] \otimes \dots \otimes k[1] \otimes \mathrm{Hom}_{\mathcal{C}}(Y_{n-1}, Y) \quad (3.2)$$

where the  $i$ -th factor  $k[1]$  is spanned by  $\varepsilon_{Y_{i+1}}$ . Here in (3.2) some of the objects  $Y_i$ s may coincide.

The differential maps  $\text{Hom}_{\mathcal{C}/\mathcal{C}_0}^{(n)}$  to  $\text{Hom}_{\mathcal{C}/\mathcal{C}_0}^{(n-1)}$ , and the category  $\mathcal{C}/\mathcal{C}_0$  is endowed with an ascending filtration.

The dg category  $\mathcal{C}/\mathcal{C}_0$  does not depend, up to a quasi-equivalence, on the choice of small dg subcategory  $\overline{\mathcal{C}_0}$ . Moreover, different choices of  $\overline{\mathcal{C}_0}$  result in *equivalent* (not just quasi-equivalent) dg categories.

It implies that we have a *functor*

$$\mathcal{P}_1\text{Cat}^{\text{dg}} \rightarrow \text{Cat}^{\text{dg}} \quad (3.3)$$

from the category  $\mathcal{P}_1\text{Cat}^{\text{dg}}$  of pairs  $(\mathcal{C}, \mathcal{C}_0)$  with  $\mathcal{C}_0$  essentially small, to the category  $\text{Cat}^{\text{dg}}$  (not just to the homotopy category  $\text{HoCat}^{\text{dg}}$ ).

### 3.3 THE DRINFELD DG QUOTIENT: A VARIATION

Let  $\mathcal{C}$  be a dg category, and let  $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$  be its full dg subcategories. Here we construct a dg category

$$\mathcal{C}/(\mathcal{C}_1, \dots, \mathcal{C}_k)$$

called *the generalized Drinfeld quotient*.

First of all, we replace the essentially small full categories  $\mathcal{C}_i$  by small full categories  $\overline{\mathcal{C}_i}$ , taking an object from each isomorphism class of objects in  $\mathcal{C}_i$ . Here we should be more careful, such that for any subset  $\{i_1, \dots, i_\ell\} \subset \{1, \dots, k\}$ , the corresponding dg categories are equivalent.

$$\overline{\mathcal{C}_{i_1}} \cap \dots \cap \overline{\mathcal{C}_{i_\ell}} \sim \mathcal{C}_{i_1} \cap \dots \cap \mathcal{C}_{i_\ell} \quad (3.4)$$

It is clear that it is always possible to achieve.

For any  $X = X_i \in \overline{\mathcal{C}_i}$ ,  $i = 1, \dots, k$ , we introduce formally an element  $\varepsilon_X^i$  which a morphism from  $X$  to  $X$  of degree -1, with  $d\varepsilon_X^i = \text{id}_X$ .

For any  $X = X_{ij} \in \overline{\mathcal{C}_i} \cap \overline{\mathcal{C}_j}$ ,  $i < j$ , we introduce formally a morphism  $\varepsilon_X^{ij}$  from  $X$  to itself of degree -2, with  $d\varepsilon_X^{ij} = \varepsilon_X^i - \varepsilon_X^j$ .

For any  $X = X^{ijk} \in \overline{\mathcal{C}_i} \cap \overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k}$ ,  $i < j < k$ , we introduce formally a morphism  $\varepsilon_X^{ijk}$  from  $X$  to itself of degree -3, with  $d\varepsilon_X^{ijk} = \varepsilon_X^{ij} - \varepsilon_X^{ik} + \varepsilon_X^{jk}$ , and so on.

The sign rule is exactly as in the Čech cohomology theory, which implies  $d^2 = 0$ .

Now the dg category  $\mathcal{C}/(\mathcal{C}_1, \dots, \mathcal{C}_k)$  has the same objects as the dg category  $\mathcal{C}$ , and the morphisms are freely generated by morphisms in  $\mathcal{C}$  with given composition among them, and the new morphisms  $\varepsilon_X^{i_1 i_2 \dots i_s}$  of degree  $-s$ , with their differentials defined as above.

Denote by  $\mathcal{C}_\Sigma$  the full dg subcategory of  $\mathcal{C}$  having the objects

$$\text{Ob}\mathcal{C}_\Sigma = \bigcup_{i=1}^k \text{Ob}\overline{\mathcal{C}_i}$$

Consider the Drinfeld dg quotient  $\mathcal{C}/\mathcal{C}_\Sigma$ . There is a natural dg functor

$$\Psi: \mathcal{C}/(\mathcal{C}_1, \dots, \mathcal{C}_k) \rightarrow \mathcal{C}/\mathcal{C}_\Sigma \quad (3.5)$$

sending all  $\varepsilon_X^i$  to  $\varepsilon_X$ , for  $X \in \text{Ob}\mathcal{C}_\Sigma$ , and sending all  $\varepsilon_X^{i_1 \dots i_s}$  to 0, for  $s > 1$ .

LEMMA 3.2. *In the above notations, the map  $\Psi$  is a quasi-equivalence of dg categories.*

*Proof.* The assertion easily follows from the two following elementary remarks.

Let  $V$  be a vector space of dimension  $n$ , with fixed basis  $\{e_1, \dots, e_n\}$ . Consider the complex  $\Lambda_d^\bullet(V)$ , having the component  $\Lambda^\ell V$  in degree  $-\ell$ , with the differential of degree  $+1$ , defined as

$$d(e_{i_1} \wedge \dots \wedge e_{i_\ell}) = \sum_{s=1}^{\ell} (-1)^{s-1} e_{i_1} \wedge \dots \wedge \hat{e}_{i_s} \wedge \dots \wedge e_{i_\ell} \quad (3.6)$$

Then the complex  $\Lambda_d^\bullet(V)$  is acyclic in all degrees.

The latter claim is clear: there is the homotopy operator  $h: \Lambda^\ell V \rightarrow \Lambda^{\ell+1} V$ ,  $h(\omega) = (e_1 + e_2 + \dots + e_n) \wedge \omega$ , with  $[d, h](\omega) = \pm \omega$ .

Consider the complex  $\Lambda_d^\bullet(U_1)$ , for  $U_1$  the 1-dimensional vector space over  $k$ , with a choused basis vector  $e$ . The natural map  $\Psi_V: \Lambda_d^\bullet(V) \rightarrow \Lambda_d^\bullet(U_1)$  which maps  $\Lambda^\ell V$  to 0 for  $\ell > 1$ , and with maps each basis vector  $e_i$  in  $V$  to the basis vector  $e$  in  $U_1$ . It is a quasi-isomorphism of (acyclic) complexes.  $\diamond$

### 3.4 THE MONOIDAL PROPERTY OF THE GENERALIZED DRINFELD DG QUOTIENT

Define the category  $\text{Cat}_-^{\text{dg}}(k)$  as the category of small  $k$ -linear dg categories  $\mathcal{C}$ , such that for any  $X, X' \in \text{Ob}\mathcal{C}$ , the graded components  $\text{Hom}^i(X, X') = 0$  for  $i > N(X, X')$ , for some integral number  $N(X, X')$  depending on  $X, X'$ .

The category  $\text{Cat}_-^{\text{dg}}(k)$  is a monoidal category. The monoidal product  $\mathcal{C} \otimes \mathcal{D}$  of two categories  $\mathcal{C}, \mathcal{D} \in \text{Cat}_-^{\text{dg}}(k)$  has objects  $\text{Ob}\mathcal{C} \times \text{Ob}\mathcal{D}$ , and

$$\text{Hom}_{\mathcal{C} \times \mathcal{D}}(X \times Y, X' \times Y') = \text{Hom}_{\mathcal{C}}(X, X') \otimes_k \text{Hom}_{\mathcal{D}}(Y, Y') \quad (3.7)$$

The r.h.s. makes sense due to the imposed vanishing condition, without it, the tensor product would be ill-defined.

Introduce the category  $\mathcal{PCat}_-^{\text{dg}}(k)$ , as follows.

An object of  $\mathcal{PCat}_-^{\text{dg}}(k)$  is a pair  $(\mathcal{C}; \mathcal{C}_1, \dots, \mathcal{C}_\ell)$ , where  $\mathcal{C} \in \text{Cat}_-^{\text{dg}}(k)$ , and the second argument is an ordered  $n$ -tuple of its full essentially small dg subcategories  $\mathcal{C}_1, \dots, \mathcal{C}_n$ , for some  $n \geq 1$ .

A morphism  $f: (\mathcal{C}; \mathcal{C}_1, \dots, \mathcal{C}_n) \rightarrow (\mathcal{D}; \mathcal{D}_1, \dots, \mathcal{D}_m)$  in  $\mathcal{PCat}_-^{\text{dg}}(k)$  requires  $m = n$ , and is given by a dg functor  $f_0: \mathcal{C} \rightarrow \mathcal{D}$ , such that  $f_0(\mathcal{C}_i) \subset \mathcal{D}_i$ , for any  $i = 1 \dots m = n$ .

The category  $\mathcal{PCat}_-^{\text{dg}}(k)$  is monoidal. Let

$$X = (\mathcal{C}; \mathcal{C}_1, \dots, \mathcal{C}_n) \text{ and } Y = (\mathcal{D}; \mathcal{D}_1, \dots, \mathcal{D}_m) \quad (3.8)$$

be two objects in  $\mathcal{PCat}_-^{\text{dg}}(k)$ . Define their product as

$$X \otimes Y = (\mathcal{C} \otimes \mathcal{D}; \mathcal{C}_1 \otimes \mathcal{D}, \dots, \mathcal{C}_n \otimes \mathcal{D}, \mathcal{C} \otimes \mathcal{D}_1, \dots, \mathcal{C} \otimes \mathcal{D}_m) \quad (3.9)$$

The monoidal product  $\otimes$  on  $\mathcal{PCat}_-^{\text{dg}}(k)$  introduced in (3.9) is strictly associative.

Now we are going to formulate the monoidal property of the Drinfeld dg quotient.

PROPOSITION 3.3. *The functor*

$$\begin{aligned} \mathcal{D}r: \mathcal{PCat}_-^{\text{dg}}(k) &\rightarrow \mathcal{Cat}_-^{\text{dg}}(k) \\ (\mathcal{C}; \mathcal{C}_1, \dots, \mathcal{C}_n) &\mapsto \mathcal{C}/(\mathcal{C}_1, \dots, \mathcal{C}_n) \end{aligned} \quad (3.10)$$

has a natural colax-monoidal structure

$$\beta_{X,Y}: \mathcal{D}r(X \otimes Y) \rightarrow \mathcal{D}r(X) \otimes \mathcal{D}r(Y) \quad (3.11)$$

with  $\beta_{X,Y}$  quasi-equivalences of dg categories, for all  $X, Y \in \mathcal{PCat}_-^{\text{dg}}(k)$ .

*Proof.* The maps  $\beta$  can be easily constructed from the following remark. Let  $V$  be a vector space of dimension  $n$ , and let  $W$  be a vector space of dimension  $m$ . Consider the (acyclic) complexes  $\Lambda_d^\bullet(V)$ ,  $\Lambda_d^\bullet(W)$ , see (3.6). Then there is a natural map of complexes

$$\beta_{V,W}: \Lambda_d^\bullet(V \oplus W) \rightarrow \Lambda_d^\bullet(V) \otimes \Lambda_d^\bullet(W)$$

which is a quasi-isomorphism (of acyclic complexes).

We constructed the dg functor  $\beta_{X,Y}: \mathcal{D}r(X \otimes Y) \rightarrow \mathcal{D}r(X) \otimes \mathcal{D}r(Y)$ . Now we construct a diagram:

$$\begin{array}{ccc} \mathcal{D}r(X \otimes Y) & \xrightarrow{\beta_{X,Y}} & \mathcal{D}r(X) \otimes \mathcal{D}r(Y) \\ & \searrow & \swarrow \\ & (\mathcal{C} \otimes \mathcal{D})/(\mathcal{C}_\Sigma \otimes \mathcal{D} \cup_f \mathcal{C} \otimes \mathcal{D}_\Sigma) & \end{array} \quad (3.12)$$

Here  $\mathcal{C}_\Sigma$  is the full dg subcategory in  $\mathcal{C}$  whose objects  $\text{Ob } \mathcal{C}_\Sigma = \text{Ob } \mathcal{C}_1 \cup \dots \cup \text{Ob } \mathcal{C}_n$ , and  $\mathcal{D}_\Sigma$  is defined analogously. The subscript  $f$  in  $\cup_f$  denotes “the full subcategory generated by the union of two parts”.

The left angle map is the map  $\Psi$  constructed in (3.5), it is a quasi-equivalence by Lemma 3.2. The right angle map is constructed in two steps.

Firstly we consider the tensor product of two  $\Psi$ -maps:

$$\Psi_{\mathcal{C}} \otimes \Psi_{\mathcal{D}} : \mathcal{C}/(\mathcal{C}_1, \dots, \mathcal{C}_n) \otimes \mathcal{D}/(\mathcal{D}_1, \dots, \mathcal{D}_m) \rightarrow \mathcal{C}/\mathcal{C}_{\Sigma} \otimes \mathcal{D}/\mathcal{D}_{\Sigma} \quad (3.13)$$

As  $\Psi_{\mathcal{C}}$  and  $\Psi_{\mathcal{D}}$  are quasi-equivalences, their tensor product also is a such one.

Secondly, we have a map

$$\Psi_2 : \mathcal{C}/\mathcal{C}_{\Sigma} \otimes \mathcal{D}/\mathcal{D}_{\Sigma} \rightarrow (\mathcal{C} \otimes \mathcal{D})/(\mathcal{C}_{\Sigma} \cup_f \mathcal{D}_{\Sigma}) \quad (3.14)$$

For construction of this map, we use the map

$$\sigma : \Lambda_d^{\bullet}(U_1) \otimes \Lambda_d^{\bullet}(U'_1) \rightarrow \Lambda_d^{\bullet}(U''_1) \quad (3.15)$$

where  $U_1, U'_1, U''_1$  are 1-dimensional vector spaces with chosen basis vectors  $e_1, e'_1, e''_1$ , correspondingly.

The map  $\sigma$  maps  $e_1 \otimes e'_1$  to 0, and it maps both  $e_1$  and  $e'_1$  to  $e''_1$ . The map  $\sigma$  is a quasi-isomorphism of (acyclic) complexes, and the corresponding map  $\Psi_2$  of dg categories is a quasi-equivalences.

Moreover, the corresponding diagram

$$\begin{array}{ccc} \Lambda_d^{\bullet}(V \oplus W) & \xrightarrow{\beta_{V,W}} & \Lambda_d^{\bullet}(V) \otimes \Lambda^{\bullet}(W) \\ & \searrow \Psi_{V \oplus W} & \downarrow \Psi_V \otimes \Psi_W \\ & & \Lambda_d^{\bullet}(U_1) \otimes \Lambda_d^{\bullet}(U'_1) \\ & & \downarrow \sigma \\ & & \Lambda_d^{\bullet}(U''_1) \end{array} \quad (3.16)$$

is commutative. Therefore, the diagram (3.12) is commutative.

As its two angle maps are quasi-equivalences, it follows from the 2-out-of-3 property of quasi-equivalences that the map  $\beta$  is a quasi-equivalence.

It remains to prove that  $\beta$  defines a colax-monoidal structure. It can be easily shown using the complexes  $\Lambda_d^{\bullet}(V)$ , and we leave this check to the reader.  $\diamond$

## 4 DELIGNE CONJECTURE FOR ESSENTIALLY SMALL ABELIAN MONOIDAL CATEGORIES

Here we prove Deligne conjecture for 1-monoidal categories in the case when the abelian category  $\mathcal{A}$  is essentially small, and has enough projective objects. The assumption that  $\mathcal{A}$  is essentially small is too strong and is not necessary, however, we decided to treat this case firstly to make

the paper more readable. In this case we have not any set-theoretical troubles in definition of the dg quotient, and this case illustrates the essential points of the construction. We hope to consider the general case, when the triangulated category  $H^0\mathcal{A}^{\text{dg}}$  is generated by a *set* of perfect objects, in our next papers.

The case when  $\mathcal{A}$  is essentially small covers the case of  $\mathcal{U}$ -generated  $C$ -modules, where  $C$  is an algebra whose underlying set is a  $\mathcal{U}$ -set. In particular, it covers the “classical” example of the category of  $A$ -bimodules, as well as the category of left modules over a bialgebra  $B$ . The case when  $\mathcal{A}$  is essentially small covers many other examples of algebraic origin.

#### 4.1 WEAK COMPATIBILITY BETWEEN THE EXACT AND THE MONOIDAL STRUCTURE IN $\mathcal{A}$

DEFINITION 4.1. Let  $\mathcal{A}$  be an abelian category, with a monoidal structure  $(\otimes, e)$  on it. Denote by  $\mathcal{A}^{\text{dg}}$  the dg category of bounded from above complexes in  $\mathcal{A}$ , and let  $\mathcal{J} \subset \mathcal{A}^{\text{dg}}$  be the full dg subcategory of acyclic objects. Consider the full additive subcategory  $\mathcal{A}_0 \subset \mathcal{A}$  where  $X \in \mathcal{A}_0$  iff  $X \otimes I$  and  $I \otimes X$  are acyclic, for any acyclic  $I \in \mathcal{J}$ . Denote by  $\mathcal{A}_0^{\text{dg}} \subset \mathcal{A}^{\text{dg}}$  the full dg subcategory of bounded from above complexes in  $\mathcal{A}_0$ . We say that the exact and the monoidal structures in  $\mathcal{A}$  are *weakly compatible* if the natural imbedding

$$\mathcal{A}_0^{\text{dg}} \hookrightarrow \mathcal{A}^{\text{dg}}$$

is a quasi-equivalence of dg categories.

LEMMA 4.2. *Let  $\mathcal{A}$  be an abelian monoidal category, whose abelian and monoidal structures are weakly compatible, see Definition above. Let  $\mathcal{J} \subset \mathcal{A}^{\text{dg}}$  be the full dg subcategory of acyclic objects, and  $\mathcal{J}_0 = \mathcal{J} \cap \mathcal{A}_0^{\text{dg}}$ . Then the natural imbedding  $\mathcal{A}_0^{\text{dg}} \hookrightarrow \mathcal{A}^{\text{dg}}$  (which is a quasi-equivalence) induces a quasi-equivalence  $\mathcal{J}_0 \hookrightarrow \mathcal{J}$ .*

*Proof.* It is enough to show that the induced map  $H^0\mathcal{J}_0 \rightarrow H^0\mathcal{J}$  of homotopy categories is essentially surjective. Let  $X \in H^0\mathcal{J}$ . We know by the assumption that the map  $H^0\mathcal{A}_0^{\text{dg}} \rightarrow H^0\mathcal{A}^{\text{dg}}$  is essentially surjective. In particular,  $X$  is isomorphic in  $H^0\mathcal{A}^{\text{dg}}$  to some object  $Y \in H^0\mathcal{A}_0^{\text{dg}}$ . The isomorphic objects in the homotopy category have isomorphic cohomology, and  $X$  is acyclic. Therefore,  $Y$  is acyclic as well, and thus  $Y \in \mathcal{J}_0$ .  $\diamond$

LEMMA 4.3. *Let  $k$  be any field, and let  $\mathcal{A}$  be an abelian  $k$ -linear category with a monoidal structure  $(\otimes, e)$ . Suppose that  $\mathcal{A}$  has enough projective objects, and  $\otimes: \mathcal{A} \times \mathcal{A} \rightarrow \text{Vect}(k)$  is a right exact bifunctor. Then the exact structure of  $\mathcal{A}$  and the monoidal structure of  $\mathcal{A}$  are weakly compatible.*

$\diamond$

## 4.2 DELIGNE CONJECTURE

Here we prove:

**THEOREM 4.4.** *Let  $\mathcal{A}$  be an abelian  $k$ -linear category, with a monoidal structure  $(\otimes, e)$ . Suppose the exact and the monoidal structure on  $\mathcal{A}$  are weakly compatible, and  $\mathcal{A}$  is essentially small. Then the graded  $k$ -vector space  $\mathrm{RHom}_{\mathcal{A}}^{\bullet}(e, e)$  has a natural structure of a Leinster 2-algebra over  $k$ , whose second product is homotopy equivalent to the Yoneda product.*

We start with a general Lemma:

**LEMMA 4.5.** *Let  $\mathcal{C}, \mathcal{D}$  be two  $k$ -linear dg categories,  $F: \mathcal{C} \rightarrow \mathcal{D}$  a dg functor which is a quasi-equivalence. Let  $\mathcal{C}_0 \subset \mathcal{C}$ ,  $\mathcal{D}_0 \subset \mathcal{D}$  be two essentially small full dg subcategories, such that  $F$  restricts to a quasi-equivalence  $F_0: \mathcal{C}_0 \rightarrow \mathcal{D}_0$ . Then the dg functor  $\overline{F}: \mathcal{C}/\mathcal{C}_0 \rightarrow \mathcal{D}/\mathcal{D}_0$  of the Drinfeld's dg quotients, is a quasi-equivalence.*

*Proof.* Let  $X, Y$  be any two objects in  $\mathcal{C}$ . We firstly prove that the map of Hom-complexes

$$\overline{F}(X, Y): \mathrm{Hom}_{\mathcal{C}/\mathcal{C}_0}(X, Y) \rightarrow \mathrm{Hom}_{\mathcal{D}/\mathcal{D}_0}(FX, FY) \quad (4.1)$$

is a quasi-isomorphism of complexes.

To this end, consider the cone  $\mathrm{Cone}(\overline{F}(X, Y))$ ; one needs to prove that it is an acyclic complex.

The complexes  $\mathrm{Hom}_{\mathcal{C}/\mathcal{C}_0}(X, Y)$ ,  $\mathrm{Hom}_{\mathcal{D}/\mathcal{D}_0}(FX, FY)$  admit natural ascending filtrations, see (3.2). The map  $\overline{F}(X, Y)$  is a map of filtered complexes. Therefore, the cone  $\mathrm{Cone}(\overline{F}(X, Y))$  inherits this filtration. We compute the corresponding to this filtration spectral sequence.

This spectral sequence  $\{E_r^{p,q}, d_r\}$  is non-zero in I and in IV quarters,  $d_r: E_r^{p,q} \rightarrow E_r^{p-r, q+r+1}$ . Therefore, it converges to  $\mathrm{gr}H^*(\mathrm{Cone}(\overline{F}(X, Y)))$  by dimensional reasons.

The term  $E_0 = \oplus_p F_{p+1}/F_p$  is easy to describe as a complex. Namely, it is a free expression (3.2), with zero values of the differential on objects  $k[1]$ . The cone of the map of complexes  $F(Y_i, Y_{i+1}): \mathcal{C}(Y_i, Y_{i+1}) \rightarrow \mathcal{D}(FY_i, FY_{i+1})$  is acyclic, as  $F$  is a quasi-equivalence. Therefore, the term  $E_1$  vanishes everywhere, and the map  $\overline{F}(X, Y)$  is a quasi-isomorphism.

It remains to prove that the corresponding map of the homotopy categories,

$$H^0\overline{F}: H^0(\mathcal{C}/\mathcal{C}_0) \rightarrow H^0(\mathcal{D}/\mathcal{D}_0) \quad (4.2)$$

is an equivalence of categories. It enough to prove  $\overline{F}$  is essentially surjective, which is clear.  $\diamond$

Now we prove the Theorem.

*Proof.* Recall the category  $\mathcal{PCat}_{-}^{\mathrm{dg}}(k)$  introduced in Section 3.4. It is a monoidal category.



Let  $\mathcal{A}$  be as in the statement of Theorem. In notations of Definition 4.1, denote  $\mathcal{J}_0 = \mathcal{A}_0 \cap \mathcal{J}$ . We construct, out of  $\mathcal{A}$ , a Segal-Leinster monoid  $F_{\mathcal{A}}$  in  $\mathcal{PCat}_{-}^{\text{dg}}(k)$ , as follows. Its underlying functor  $F_{\mathcal{A}}: \Delta_f^{\text{opp}} \rightarrow \mathcal{PCat}_{-}^{\text{dg}}(k)$  is defined on objects as

$$F_{\mathcal{A}}([n]) = (\mathcal{A}_0^{\otimes n}; \mathcal{C}_1^{[n]}, \dots, \mathcal{C}_n^{[n]}) \quad (4.3)$$

where

$$\begin{aligned} \mathcal{C}_1^{[n]} &= \mathcal{J}_0 \otimes \mathcal{A}_0 \otimes \mathcal{A}_0 \otimes \dots \otimes \mathcal{A}_0 \quad \text{totally } n-1 \text{ factors } \mathcal{A}_0 \\ \mathcal{C}_2^{[n]} &= \mathcal{A}_0 \otimes \mathcal{J}_0 \otimes \mathcal{A}_0 \otimes \dots \otimes \mathcal{A}_0 \quad \text{totally } n-1 \text{ factors } \mathcal{A}_0 \\ &\dots\dots\dots \\ \mathcal{C}_n^{[n]} &= \mathcal{A}_0 \otimes \mathcal{A}_0 \otimes \dots \otimes \mathcal{A}_0 \otimes \mathcal{J}_0 \quad \text{totally } n-1 \text{ factors } \mathcal{A}_0 \end{aligned} \quad (4.4)$$

(for the dg category  $\mathcal{C}_i^{[n]}$ , the factor  $\mathcal{J}_0$  is at  $i$ -th place).

The claim that  $F_{\mathcal{A}}: \Delta_f^{\text{opp}} \rightarrow \mathcal{PCat}_{-}^{\text{dg}}(k)$  is a functor, follows from the fact the  $\mathcal{J}_0$  is a two-sided ideal in  $\mathcal{A}_0$ :  $\mathcal{A}_0 \odot \mathcal{J}_0 \subset \mathcal{J}_0$ ,  $\mathcal{J}_0 \odot \mathcal{A}_0 \subset \mathcal{J}_0$ , which follows from Definition 4.1. Then we deduce that the Segal-Leinster monoid defined out of a strict monoidal category  $\mathcal{A}_0$  as in Definition 2.2, descends to a Segal-Leinster monoid structure on the quotients, with the identity colax-maps  $F_{\mathcal{A}}([m+n]) \rightarrow F_{\mathcal{A}}([m]) \otimes F_{\mathcal{A}}([n])$ .

Next, we apply the generalized Drinfeld dg quotient functor  $\mathcal{D}r: \mathcal{PCat}_{-}^{\text{dg}}(k) \rightarrow \mathcal{Cat}_{-}^{\text{dg}}(k)$ , defined in Proposition 3.3. By this Proposition, the functor  $\mathcal{D}r$  has a natural colax-monoidal structure, whose colax maps are quasi-equivalences of dg categories.

We have the composition of colax-monoidal functors:

$$\Delta_f^{\text{opp}} \xrightarrow{F_{\mathcal{A}}} \mathcal{PCat}_{-}^{\text{dg}}(k) \xrightarrow{\mathcal{D}r} \mathcal{Cat}_{-}^{\text{dg}}(k) \quad (4.5)$$

As a composition of such ones, the total functor is a colax-monoidal functor, whose colax maps are quasi-equivalences.

Next, we notice that for each  $[n]$ , the category  $\mathcal{D}r \circ F_{\mathcal{A}}[n]$  has a distinguished object  $*_{[n]}$ . In fact,

$$*_{[n]} = \mathcal{D}r(e \otimes e \otimes \dots \otimes e) \quad n \text{ factors } e \quad (4.6)$$

where  $e$  is the unit in  $\mathcal{A}$  (which belongs to  $\mathcal{A}_0$  and is the unit in it). It means that the image of composition (4.6) takes values in the category  $\mathcal{Cat}_{-*}^{\text{dg}}(k)$  of the corresponding dg categories *with a marked object*.

There is a natural functor

$$\mathcal{H}om: \mathcal{Cat}_{-*}^{\text{dg}}(k) \rightarrow \mathcal{Mon}(k), \quad \mathcal{C} \mapsto \text{Hom}_{\mathcal{C}}(*, *) \quad (4.7)$$

to the category of monoids in  $\mathcal{Vect}(k)$  (that is, to the category of associative dg algebras over  $k$ ).

The functor  $\mathcal{H}om$  is colax-monoidal with the identity colax maps.  
Now the composition of (4.6) with the functor  $\mathcal{H}om$  gives a functor

$$\mathcal{H}om \circ \mathcal{D}r \circ F_{\mathcal{A}} : \Delta_0^{\text{opp}} \rightarrow \text{Mon}(k) \quad (4.8)$$

As a composition of such ones, it is a colax-monoidal functor, whose colax maps are quasi-isomorphisms of dg algebras over  $k$ .

KEY-LEMMA 4.6.  $\mathcal{H}om \circ \mathcal{D}r \circ F_{\mathcal{A}}([1]) = \text{RHom}_{\mathcal{A}}^{\bullet}(e, e)$

*Proof.* The composition in the statement of Key-Lemma is the Hom-complex  $(\mathcal{A}_0/\mathcal{J}_0)(e, e)$ . By Lemmas 4.5 and 4.2, we know that  $\mathcal{A}_0/\mathcal{J}_0$  is quasi-equivalent to  $\mathcal{A}/\mathcal{J}$ . Now the claim follows from Proposition 3.1, which gives the Keller's description of the dg quotient for a pre-triangulated dg category  $\mathcal{C}$  and its full pre-triangulated dg subcategory  $\mathcal{N}$ .  $\diamond$

So what we get is the following. We constructed a Segal-Leinster monoid  $\mathcal{F}_{\mathcal{A}}$  in the monoidal category of dg algebras over  $k$ , whose component  $\mathcal{F}_{\mathcal{A}}[1]$  is quasi-isomorphic to  $\text{RHom}_{\mathcal{A}}^{\bullet}(e, e)$ . The higher components  $\mathcal{F}_{\mathcal{A}}[n]$ ,  $n > 1$ , are hard to compute, but we do not need to know them explicitly. Only what we need is that for *some* higher components, the dg algebra  $\text{RHom}_{\mathcal{A}}^{\bullet}(e, e)$  can be complemented (as its first component) to a Segal-Leinster monoid in the monoidal category of dg algebras over  $k$ . It means, by Definition 2.5, that  $\text{RHom}_{\mathcal{A}}^{\bullet}(e, e)$  is a Leinster 2-algebra.  $\diamond$

## 5 DELIGNE CONJECTURE FOR ESSENTIALLY SMALL $n$ -FOLD MONOIDAL ABELIAN CATEGORIES

We suppose some familiarity with the basic definitions and facts of [BFSV, Section 1]. We will be rather sketchy here, pointing out on what is essentially new respectively with  $n = 1$  case.

### 5.1 FORMULATION OF THE RESULT

DEFINITION 5.1. Let  $\mathcal{A}$  be an abelian category, with an  $n$ -fold monoidal structure  $(\otimes_1, \dots, \otimes_n; e)$  on it. Denote by  $\mathcal{A}^{\text{dg}}$  the dg category of bounded from above complexes in  $\mathcal{A}$ , and let  $\mathcal{J} \subset \mathcal{A}^{\text{dg}}$  be the full dg subcategory of acyclic objects. Consider the full additive subcategory  $\mathcal{A}_0 \subset \mathcal{A}$  where  $X \in \mathcal{A}_0$  iff  $X \otimes_i J$  and  $J \otimes_i X$  are acyclic, for any acyclic  $J \in \mathcal{J}$ , and for any  $1 \leq i \leq n$ . Denote by  $\mathcal{A}_0^{\text{dg}} \subset \mathcal{A}^{\text{dg}}$  the full dg subcategory of bounded from above complexes in  $\mathcal{A}_0$ . We say that the exact and the monoidal structures in  $\mathcal{A}$  are *weakly compatible* if the natural imbedding

$$\mathcal{A}_0^{\text{dg}} \hookrightarrow \mathcal{A}^{\text{dg}}$$

is a quasi-equivalence of dg categories.

The next definition did not appear in our discussion of the case  $n = 1$ .

**DEFINITION 5.2.** Let  $\mathcal{A}$  be an abelian  $k$ -linear category, with a weakly compatible  $n$ -fold monoidal structure on it. We say that the  $n$ -fold monoidal structure is *non-degenerate* if for any four objects  $X, Y, Z, W$  in  $\mathcal{A}_0^{\text{dg}}$ , and for any  $1 \leq i < j \leq n$ , the  $(i, j)$ -th Eckmann-Hilton map  $\eta_{ij}: (X \otimes_j Y) \otimes_i (Z \otimes_j W) \rightarrow (X \otimes_i Z) \otimes_j (Y \otimes_i W)$  becomes an isomorphism in the homotopy category  $H^0(\mathcal{A}_0^{\text{dg}})$ .

Now we are ready to formulate our version of Deligne conjecture for  $n$ -fold monoidal abelian categories.

**THEOREM 5.3.** *Let  $k$  be a field of characteristic 0, and let  $\mathcal{A}$  be essentially small  $k$ -linear abelian category, endowed with  $k$ -linear  $n$ -fold monoidal structure. Suppose the abelian and the  $n$ -fold monoidal structures are weakly compatible, see Definition 5.1, and suppose that the  $n$ -fold monoidal structure is non-degenerate, see Definition 5.2. Then  $\text{RHom}_{\mathcal{A}}^{\bullet}(e, e)$  has a natural structure of a Leinster  $(n+1)$ -algebra over  $k$ , whose underlying Leinster 1-algebra is homotopy equivalent to the Yoneda product.*

**REMARK 5.4.** It is very important to notice that validity of the Deligne conjecture requires the non-degeneracy assumption for the Eckmann-Hilton maps  $\eta_{ij}$ , see Definition 5.2. This assumption appears when we apply the Street rectification theorem, see below. Note that for a general  $n$ -fold monoidal category in sense of [BFSV], the maps  $\eta_{ij}$  may be *arbitrary* maps, not necessarily (quasi-)isomorphisms.

Below we sketch a proof of Theorem 5.3.

## 5.2 STREET RECTIFICATION AND VARIATIONS

Here we recall what the Street rectification [Str] is, as it is essential for the proof of Theorem 5.3.

Denote by  $\mathcal{Cat}$  the category of small set-enriched categories. As  $\mathcal{Cat}$  is a 2-category, one can speak on lax and on colax functors  $\mathcal{I} \rightarrow \mathcal{Cat}$ .

Suppose  $F: \mathcal{I} \rightarrow \mathcal{Cat}$  be a lax or a colax functor. It means essentially that for two morphisms  $\alpha \in \text{Hom}_{\mathcal{I}}(X, Y)$  and  $\beta \in \text{Hom}_{\mathcal{I}}(Y, Z)$  the genuine functor equality  $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$  fails, but instead there are maps

$$\omega_{\alpha\beta}: F(\beta) \circ F(\alpha) \rightarrow F(\beta \circ \alpha)$$

in the case of a lax functor  $F$ , or

$$\omega_{\alpha\beta}: F(\beta \circ \alpha) \rightarrow F(\beta) \circ F(\alpha)$$

in the case of a colax functor  $F$ .

In both cases,  $F(\beta \circ \alpha)$  and  $F(\beta) \circ F(\alpha)$  are functors from the category  $F(X)$  to the category  $F(Z)$ , and  $\omega_{\alpha\beta}$  are natural transformations of functors. Here the 2-category structure on  $\mathcal{Cat}$  is used.

These  $\omega_{\alpha\beta}$  are subject to some natural relations, for which we refer the reader to, e.g., [Th, Section 3].

The Street rectification is the following result.

**PROPOSITION 5.5** (Street rectification). *Let  $I$  be an ordinary category, and  $F: I \rightarrow \mathcal{Cat}$  be a lax functor (correspondingly, a colax functor). Then there exists a genuine functor  $\hat{F}: I \rightarrow \mathcal{Cat}$ , such that for any  $i \in I$  there is an adjoint pair of functors*

$$p: \hat{F}(i) \rightleftarrows F(i) : q \quad (5.1)$$

*functorial for morphisms  $i \rightarrow i'$  in  $I$ .*

See [Str] for a proof. ◇

In the context of dg categories, we have the concept of a quasi-equivalence dg functor. It is a property for a 1-morphism in  $\mathcal{Cat}^{\text{dg}}(k)$ . We extend it to a property of 2-morphisms (natural transformations of functors).

**DEFINITION 5.6.** A natural transformation  $\omega: F \Rightarrow G$  between two dg functors  $F, G: \mathcal{C} \rightarrow \mathcal{D}$ ,  $D \in \mathcal{Cat}^{\text{dg}}(k)$  is called a *homotopy equivalence* if for any  $X \in \mathcal{C}$ , the corresponding morphism  $\omega(X): F(X) \rightarrow G(X)$  is a closed morphism, and it becomes an isomorphism in the homotopy category  $H^0(\mathcal{D})$ .

We formulate two specifications of the Street rectification theorem in dg setting, which can be proven by elementary application of constructions in [Str].

**PROPOSITION 5.7.** *Suppose we have a (co)lax functor  $F: I \rightarrow \mathcal{Cat}^{\text{dg}}(k)$ , such that its colax maps  $\omega_{\alpha\beta}$  are homotopy equivalences, see Definition 5.6, for any composable morphisms  $\alpha, \beta$  in  $I$ . Then there exists a rectification  $\hat{F}$  of  $F$ , such the adjoint natural transformations  $p, q$  in (5.1) are both homotopy equivalences.* ◇

We need a version of this result for a colax-monoidal colax functors.

**PROPOSITION 5.8.**

- (1) *Let  $I$  be a monoidal category, and let  $F: I \rightarrow \mathcal{Cat}^{\text{dg}}(k)$  be a colax functor with a colax-monoidal structure on it. Then the genuine functor  $\hat{F}$  is also a colax monoidal functor with colax-monoidal maps  $\beta_{ij}: F(i * j) \rightarrow F(i) \otimes_k F(j)$ ,  $i, j \in I$*

- (2) Suppose that all colax functor maps  $\omega_{\alpha\beta}$  of  $F$  are homotopy equivalences, and suppose that all colax-monoidal maps  $\beta$  are homotopy equivalences between the two corresponding functors  $I \times I: \mathcal{Cat}^{\text{dg}}(k)$ . Then the colax-monoidal map  $\hat{\beta}$  of corresponding genuine colax-monoidal functor  $\hat{F}$ , is a homotopy equivalence (as a natural transformation between the functors  $I \times I \rightarrow \mathcal{Cat}^{\text{dg}}(k)$ ).

◇

### 5.3 PROOF OF THEOREM 5.3

We start with some general remarks.

Let  $\mathcal{C}$  be a set-enriched  $n$ -fold monoidal category. It proven in [BFSV, Theorem 2.1] that the assignment

$$F([n_1] \times \cdots \times [n_\ell]) = \mathcal{C}^{\times(n_1 \dots n_\ell)} \quad (5.2)$$

defines a colax functor

$$F: (\Delta^{\text{opp}})^{\times n} \rightarrow \mathcal{Cat} \quad (5.3)$$

which obeys the Segal monoidal property.

Then it is applied the Street rectification theorem in [BFSV, Section 2], to replace the colax functor  $F$  by a genuine functor  $\hat{F}$ .

When  $\mathcal{C}$  is a  $k$ -linear category  $n$ -fold monoidal category, and  $G$  is defined as

$$G([n_1] \times \cdots \times [n_\ell]) = \mathcal{C}^{\otimes_k(n_1 \dots n_\ell)} \quad (5.4)$$

it fails to be a functor from  $(\Delta^{\text{opp}})^{\times n} \rightarrow \mathcal{Cat}(k)$  (where  $\mathcal{Cat}(k)$  stands for the category of small  $k$ -linear categories). However, one has:

**PROPOSITION 5.9.** *The functor  $G$  is a colax-functor  $G: (\Delta_f^{\text{opp}})^{\times n} \rightarrow \mathcal{Cat}(k)$ , endowed with a colax-monoidal structure*

$$\beta_{xy}: G(x \times y) \rightarrow G(x) \otimes_k G(y)$$

*with the equivalence colax-maps  $\beta_{x,y}$  (here  $x, y \in (\Delta_f^{\text{opp}})^{\times n}$ , and  $x \odot y$  denote the monoidal product in  $(\Delta_f^{\text{opp}})^{\times n}$ ).*

It follows immediately from the proof of Theorem 2.1 in [BFSV].

◇

However, when we work with  $k$ -linear dg categories, it is not true, in general, that  $G$  and  $\hat{G}$  define the same functors  $(\Delta_f^{\text{opp}})^n \rightarrow \text{HoCat}^{\text{dg}}(k)$  to the homotopy category of dg categories. It is the case when the adjoint maps  $p$  and  $q$  in (5.1) are homotopy equivalences, see Definition 5.6. This is guaranteed by the non-degeneracy condition. We have:

**LEMMA 5.10.** *Let  $\mathcal{B}$  be an  $n$ -fold monoidal dg category, with the non-degeneracy condition, see Definition 5.2. Then the colax-functor maps  $\omega_{\alpha\beta}$  of the corresponding colax-functor  $G$ , see (5.4), are homotopy equivalences, see Definition 5.6.*

It follows from the construction of  $G$  and from its investigation in [BFSV, Section 2].

◇

COROLLARY 5.11. *Within the assumptions of Lemma 5.10, the corresponding adjoint pair of natural transformations (5.1),*

$$p: \hat{G} \rightleftarrows G : q$$

*are homotopy equivalences in the sense of Definition 5.6. In particular,  $p$  and  $q$  are invertible in the homotopy category of dg categories  $\mathrm{HoCat}^{\mathrm{dg}}(k)$ .*

*Proof.* It follows from Lemma 5.10 and Proposition 5.7.

◇

After these remarks, we pass to the proof properly.

Let  $\mathcal{A}$  be a  $k$ -linear abelian category with  $n$ -fold monoidal structure on it. Suppose that the abelian and the  $n$ -fold monoidal structures are weakly compatible, see Definition 5.1, and suppose that the corresponding  $n$ -fold monoidal dg category  $\mathcal{A}_0^{\mathrm{dg}}$  is non-degenerate, see Definition 5.2.

Define the corresponding to the  $n$ -fold monoidal category  $\mathcal{A}_0^{\mathrm{dg}}$  colax-monoidal colax functor

$$G: (\Delta_f^{\mathrm{opp}})^{\times n} \rightarrow \mathrm{Cat}_-^{\mathrm{dg}}(k) \quad (5.5)$$

$$G([n_1] \times \cdots \times [n_\ell]) = (\mathcal{A}_0^{\mathrm{dg}})^{\otimes (n_1, \dots, n_\ell)} \quad (5.6)$$

We skip here the replacement of the essentially small dg categories by their small dg subcategories, as was done in Section 3.4. We use the same notations for the corresponding small dg categories.

Now we modify the colax functor  $G: (\Delta_f^{\mathrm{opp}})^{\times n} \rightarrow \mathrm{Cat}_-^{\mathrm{dg}}(k)$ , to a colax functor  $F: (\Delta_f^{\mathrm{opp}})^{\times n} \rightarrow \mathcal{P}\mathrm{Cat}_-^{\mathrm{dg}}(k)$ , as in our proof of  $n = 1$  case, see (4.3), (4.4). More precisely, denote  $\mathcal{J}_0 = \mathcal{A}_0 \cap \mathcal{J}$ , see Definition 5.1. Then

$$F([n_1] \times [n_2] \times \cdots \times [n_\ell]) = \left( (\mathcal{A}_0^{\mathrm{dg}})^{\otimes (n_1, \dots, n_\ell)}; \{ \mathcal{C}_{i_1, \dots, i_\ell}^{[n_1, \dots, n_\ell]} \}_{1 \leq i_1 \leq n_1, \dots, 1 \leq i_\ell \leq n_\ell} \right) \quad (5.7)$$

with

$$\mathcal{C}_{i_1, \dots, i_\ell}^{[n_1, \dots, n_\ell]} = \mathcal{C}_{i_1}^{[n_1]} \otimes_k \mathcal{C}_{i_2}^{[n_2]} \otimes \cdots \otimes \mathcal{C}_{i_\ell}^{[n_\ell]} \quad (5.8)$$

where  $\mathcal{C}_{i_k}^{[n_k]}$  are defined as in (4.4).

Recall that  $\mathcal{P}\mathrm{Cat}_-^{\mathrm{dg}}$  is a monoidal category of tuples of dg categories, see Section 3.4.

Nextly, we replace the colax functor  $F$  by a genuine functor  $\hat{F}$ , using the Street rectification.

Then what follows is fairly analogous to our proof of Deligne conjecture for  $n = 1$ , see Section 4.2.

The functor  $\hat{F}$  is colax-monoidal with (quasi-)equivalences colax maps. What is very important, the functor  $\hat{F}$  and the colax functor  $F$  are related by a pair of adjoint natural transformations (5.1), where both  $p, q$  are homotopy equivalences, by Corollary 5.11.

We construct the composition

$$(\Delta_f^{\text{opp}})^{\times n} \xrightarrow{\hat{F}} \mathcal{P}\mathcal{C}at_-^{\text{dg}}(k) \xrightarrow{\mathcal{D}r} \mathcal{C}at_-^{\text{dg}}(k) \xrightarrow{\mathcal{H}om} \mathcal{M}on(k) \quad (5.9)$$

where  $\mathcal{D}r$  is the modified Drinfeld dg quotient, constructed in Sections 3.3, 3.4, and  $\mathcal{H}om(\mathcal{C}) = \text{Hom}_{\mathcal{C}}(*, *)$ . (Here, as well as in the proof of Theorem 4.4, we use the fact that the categories  $\mathcal{D}r \circ \hat{F}([n_1] \times \cdots \times [n_\ell])$  have “based” objects  $*$ , which are the images of  $[0] \times \cdots \times [0]$  by the unique degeneracy morphism  $[n_1] \times \cdots \times [n_\ell] \rightarrow [0] \times \cdots \times [0]$ ).

KEY-LEMMA 5.12. *The composition*

$$\mathcal{H}om \circ \mathcal{D}r \circ \hat{F}([1] \times \cdots \times [1]) \simeq \text{RHom}_{\mathcal{A}}^\bullet(e, e)$$

We only need to check that  $\hat{F}([1] \times [1] \times \cdots \times [1]) \simeq \mathcal{A}_0$  (where  $\simeq$  stands for “quasi-equivalent”). This is guaranteed by our assumption that the  $n$ -fold monoidal category  $\mathcal{A}$  is non-degenerate, see Definition 5.2, and by Corollary 5.11. The rest is analogous to the proof of Key-Lemma 4.6.

◇

Theorem 5.3 is proven.

## 6 AN APPLICATION: THE GERSTENHABER-SCHACK COMPLEX OF A HOPF ALGEBRA

For any associative bialgebra  $B$  over  $k$ , there is a concept of a *tetramodule* over  $B$ . Tetramodules form an abelian  $k$ -linear category, denoted by  $\mathcal{T}etra(B)$ . We proved in [Sh1], [Sh3] that  $\mathcal{T}etra(B)$  has a natural structure of a 2-fold monoidal category, in sense of [BFSV]. Here we prove the following

THEOREM 6.1. *Suppose  $B$  is a Hopf algebra over  $k$ . Then the 2-fold monoidal abelian category  $\mathcal{T}etra(B)$  satisfies the assumptions of Definitions 5.1 and 5.2. In particular, Theorem 5.3 is applicable to this category. In particular,  $\text{RHom}_{\mathcal{T}etra(B)}^\bullet(B, B)$  has a structure of a Leinster 3-algebra over  $k$ .*

### 6.1 DEFINITIONS

Recall that an *associative bialgebra* over  $k$  is a  $k$ -vector space  $B$ , endowed with a product  $B \otimes B \rightarrow B$ , a coproduct  $\Delta: B \rightarrow B \otimes B$ , a unit  $i: k \rightarrow B$ , a counit  $\varepsilon: B \rightarrow k$  such that:

- (i)  $(m, i)$  defines the structure of an associative algebra with unit on  $B$ ,
- (ii)  $(\Delta, \varepsilon)$  defines a structure of a coassociative coalgebra with counit on  $B$ ,

(iii) the compatibility:  $\Delta(a * b) = \Delta(a) * \Delta(b)$  (here  $a * b = m(a, b)$ ),

(iv) the counit is an algebra map, the unit is a coalgebra map.

Recall that an associative bialgebra is called a *Hopf algebra*, if there exists a  $k$ -linear *antipode* map  $S: B \rightarrow B$ , satisfying the following properties:

(i)  $S$  is a linear isomorphism,

(ii)  $m(1 \otimes S)\Delta(x) = m(S \otimes 1)\Delta(x) = i(\varepsilon(x))$

One can deduce from this definition that

(iii)  $S(a * b) = S(b) * S(a)$ ,  $\Delta(S(x)) = (S \otimes S)(\Delta^{\text{op}}(x))$ ,

(iv)  $\varepsilon(S(x)) = \varepsilon(x)$ ,  $S(i(1)) = i(1)$ .

Recall that a *tetramodule* over a bialgebra  $B$  is a  $k$ -vector space  $M$  such that  $(B \oplus \epsilon M)[\epsilon]/(\epsilon^2)$  is once again an associative bialgebra, over  $k[\epsilon]/(\epsilon^2)$ , such that the canonical maps  $B[\epsilon]/(\epsilon^2) \rightarrow (B \oplus \epsilon M)[\epsilon]/(\epsilon^2)$  and  $(B \oplus \epsilon M)[\epsilon]/(\epsilon^2) \rightarrow B[\epsilon]/(\epsilon^2)$  are bialgebra maps. It results to four structures:

T(i) a left  $B$ -module structure  $m_\ell: B \otimes M \rightarrow M$ ,

T(ii) a right  $B$ -module structure  $m_r: M \otimes B \rightarrow B$ ,

T(iii) a left comodule structure  $\Delta_\ell: M \rightarrow B \otimes M$ ,

T(iv) a right comodule structure  $\Delta_r: M \rightarrow M \otimes B$

subject to the following compatibilities:

TC(i) equipped with  $(m_\ell, m_r)$ ,  $M$  is a bimodule,

TC(ii) equipped with  $(\Delta_\ell, \Delta_r)$ ,  $M$  is a bi-comodule,

TC(iii) four “bialgebra compatibilities”:

$$\Delta_\ell(a * m) = (\Delta^1(a) * \Delta_\ell^1(m)) \otimes (\Delta^2(a) * \Delta_\ell^2(m)) \subset B \otimes_k M \quad (6.1)$$

$$\Delta_\ell(m * a) = (\Delta_\ell^1(m) * \Delta^1(a)) \otimes (\Delta_\ell^2(m) * \Delta^2(a)) \subset B \otimes_k M \quad (6.2)$$

$$\Delta_r(a * m) = (\Delta^1(a) * \Delta_r^1(m)) \otimes (\Delta^2(a) * \Delta_r^2(m)) \subset M \otimes_k B \quad (6.3)$$

$$\Delta_r(m * a) = (\Delta_r^1(m) * \Delta^1(a)) \otimes (\Delta_r^2(m) * \Delta^2(a)) \subset M \otimes_k B \quad (6.4)$$



As well, we see that the underlying  $k$ -vector space  $B$  is a  $B$ -tetramodule. It is the unit of the two-fold monoidal structure on  $\mathcal{Tetra}(B)$ , constructed in [Sh1], [Sh3].

The category of tetramodules is very important because of its relation to the Gerstenhaber-Schack complex [GS], the “deformation complex” of an associative bialgebra. The following result is due to R. Taillefer [Ta1,2] (see also an overview of Taillefer’s results in [Sh1, Section 3]).

**PROPOSITION 6.2.** *Let  $B$  be an associative algebra over a field  $k$ . Then the Gerstenhaber-Schack complex of  $B$  is quasi-isomorphic to  $\mathrm{RHom}_{\mathcal{Tetra}(B)}^\bullet(B, B)$ .*

◇

## 6.2 A PROOF OF THEOREM 6.1

Recall that a Hopf module over a bialgebra  $B$  over  $k$  is a  $k$ -vector spaces  $M$ , endowed with a left  $B$ -module structure  $m_\ell: B \otimes M \rightarrow M$ , with a left  $B$ -comodule structure  $\Delta_\ell: M \rightarrow B \otimes M$ , such that (6.1) holds. In particular, any tetramodule over  $B$  defines an underlying Hopf module over  $B$ .

The proof of Theorem 6.1 is based on the following classical result, see [Sw, Theorem 4.1.1]:

**KEY-LEMMA 6.3.**

- (i) *Let  $B$  be a Hopf algebra, and let  $M$  be a left Hopf module over  $B$ . Then  $M$  is free as left  $B$ -module and is cofree as left  $B$ -comodule. An analogous claim is true also when  $M$  is a right Hopf module over  $B$ . More specifically, let  $M$  be a left Hopf module over  $B$ , denote by  $\Delta_\ell: M \rightarrow B \otimes M$  its comodule map. Denote*

$$M_\Delta = \{m \in M \mid \Delta_\ell(m) = 1 \otimes m\}$$

*Then the map of left  $B$ -modules*

$$\alpha: B \otimes_k M_\Delta \rightarrow M$$

*$b \otimes m' \mapsto b \cdot m'$ , is an isomorphism of Hopf modules. Analogously for right Hopf modules.*

- (ii) *The inverse to the map  $\alpha$  is a map*

$$\beta: M \rightarrow B \otimes M_\Delta$$

*is given by*

$$\beta(m) = (\mathrm{id}_B \otimes P) \circ \Delta_\ell(m) \tag{6.5}$$

*where  $P$  is the composition*

$$P: M \xrightarrow{\Delta_\ell} B \otimes M \xrightarrow{S \otimes \mathrm{id}_M} B \otimes M \xrightarrow{m_\ell} M \tag{6.6}$$

*and, in fact,*

$$P(M) = M_\Delta \tag{6.7}$$

◇

REMARK 6.4. In Key-Lemma above, it is very essential that  $B$  is a Hopf algebra. The claim fails when  $B$  is a general associative bialgebra.

Let  $B$  be a Hopf algebra over  $k$ . Consider the bialgebra  $B^{\text{opp}}$ , whose underlying vector space is  $B$ , and the product and the coproduct are opposite to those in  $B$ :  $m_{B^{\text{opp}}}(a \otimes b) = m_B(b \otimes a)$ ,  $\Delta_{B^{\text{opp}}}(x) = \sigma_{12}(\Delta_B(x))$  (where  $\sigma_{12}: B \otimes B \rightarrow B \otimes B$  is the permutation of the factors). Then  $B^{\text{opp}}$  is a Hopf algebra as well, with the same antipode  $S_{B^{\text{opp}}} = S_B$ .

Let  $C, D$  be Hopf algebras over  $k$ . Then their tensor product  $C \otimes_k D$  is naturally a Hopf algebra, with the antipode  $S_{C \otimes D} = S_C \otimes S_D$ .

Therefore, we have:

LEMMA 6.5. *Let  $B$  be a Hopf algebra over  $k$  with antipode  $S$ . Then  $B \otimes_k B^{\text{opp}}$  is naturally a Hopf algebra, with antipode  $S \otimes S$ .*

◇

LEMMA 6.6. *Let  $B$  be an associative bialgebra. Then a tetramodule over  $B$  is the same that a left Hopf bimodule over  $B \otimes B^{\text{opp}}$ .*

It is clear.

◇

COROLLARY 6.7. *Let  $B$  be a Hopf algebra over  $k$ . Then any tetramodule  $M$  over  $B$  has the following description. Consider a  $k$ -vector subspace  $M_0 \subset M$  defined as*

$$M_0 = \{m \in M \mid \Delta_\ell(m) = 1 \otimes m, \Delta_r(m) = m \otimes 1\}$$

*Then  $B \otimes_k B^{\text{opp}} \otimes_k M_0$  is naturally a tetramodule, and the map*

$$B \otimes B^{\text{opp}} \otimes M_0 \rightarrow M, \quad b \otimes b' \otimes m_0 \mapsto b \cdot m_0 \cdot b' \quad (6.8)$$

*is an isomorphism of tetramodules.*

*Proof.* It follows from Key-Lemma 6.3, Lemma 6.5, and Lemma 6.6.

◇

We pass to a proof of Theorem 6.1.

Theorem 6.1 follows easily from the following result.

PROPOSITION 6.8. *Let  $B$  be a Hopf algebra over  $k$ . Then both monoidal products  $\otimes_1$  and  $\otimes_2$  forming the 2-fold monoidal structure on  $\mathcal{Tetra}(B)$  are exact bifunctors, and the Eckmann-Hilton map*

$$\eta(MNPQ): (M \otimes_2 N) \otimes_1 (P \otimes_2 Q) \rightarrow (M \otimes_1 P) \otimes_2 (N \otimes_1 Q)$$

*is an isomorphism, for any four  $B$ -tetramodules  $M, N, P, Q$ .*

Indeed, assuming the Proposition holds, one has  $\mathcal{A}_0 = \mathcal{A} = \mathcal{Tetra}(B)$  (which fulfill the assumption from Definition 5.1), and the assumption from Definition 5.2 is fulfilled as the Eckmann-Hilton map is an isomorphism for any 4 tetramodules.

It remains to prove Proposition 6.8.

It follows from Key-Lemma 6.3 that each  $B$ -tetramodule  $M$  is a free left  $B$ -module, a free right  $B$ -module, a cofree left  $B$ -comodule, and a cofree right  $B$ -comodule. Then it follows from our construction of  $\otimes_1$  and  $\otimes_2$  that each of these two bifunctors is bi-exact on  $\mathcal{Tetra}(B)$ . This proves the first statement of Proposition 6.8.

For the second statement, we need to recall the constructions of the two monoidal products  $\otimes_1$  and  $\otimes_2$ , along with the Eckmann-Hilton map  $\eta_{MNPQ}$ , on the category  $\mathcal{Tetra}(B)$ . Then we compute all of them explicitly in the case when  $B$  is a Hopf algebra. It is done in Subsection 6.3 below.

### 6.3 THE 2-FOLD MONOIDAL STRUCTURE ON $\mathcal{Tetra}(B)$

Recall some constructions of [Sh1]. Let  $B$  be an associative bialgebra, and let  $M, N$  be two tetramodules over it.

One firstly define two their “external” tensor products  $M \boxtimes_1 N$  and  $M \boxtimes_2 N$  (which are  $B$ -tetramodules once again). In both cases the underlying vector space is  $M \otimes_k N$ . The tetramodule structures are defined as follows (where  $a \in B$ ,  $m \in M$ ,  $n \in N$ ):

*The case of  $M \boxtimes_1 N$ :*

1.  $m_\ell(a \otimes m \boxtimes n) = (am) \boxtimes n$ ,
2.  $m_r(m \boxtimes n \otimes a) = m \boxtimes (na)$ ,
3.  $\Delta_\ell(m \boxtimes n) = (\Delta_\ell^1(m) * \Delta_\ell^1(n)) \otimes (\Delta_\ell^2(m) \boxtimes \Delta_\ell^2(n))$ ,
4.  $\Delta_r(m \boxtimes n) = (\Delta_r^1(m) \boxtimes \Delta_r^1(n)) \otimes (\Delta_r^2(m) * \Delta_r^2(n))$ .

*The case of  $M \boxtimes_2 N$ :*

1.  $m_\ell(a \otimes m \boxtimes n) = (\Delta^1(a)m) \boxtimes (\Delta^2(a)n)$ ,
2.  $m_r(m \boxtimes n \otimes a) = (m\Delta^1(a)) \boxtimes (n\Delta^2(a))$ ,
3.  $\Delta_\ell(m \boxtimes n) = \Delta_\ell^1(m) \otimes (\Delta_\ell^2(m) \boxtimes n)$ ,
4.  $\Delta_r(m \boxtimes n) = (m \boxtimes \Delta_r^1(n)) \otimes \Delta_r^2(n)$ .

Next, one defines

$$M \otimes_1 N = M \boxtimes_1 N / \left\{ \sum_i (m_i a) \boxtimes_1 n_i - \sum_i m_i \boxtimes_1 (a n_i), a \in B \right\} \quad (6.9)$$

and

$$M \otimes_2 N = \left\{ \sum_i m_i \boxtimes_2 n_i \in M \boxtimes_2 N \mid \sum_i \Delta_r(m_i) \otimes_k n_i = \sum_i m_i \otimes_k \Delta_\ell(n_i) \right\} \quad (6.10)$$

LEMMA 6.9. *Let  $B$  be a Hopf algebra. Decompose, as in Corollary 6.7,  $M = B \otimes B^{\text{opp}} \otimes M_0$ ,  $N = B \otimes B^{\text{opp}} \otimes N_0$ . Then the two monoidal products  $M \otimes_1 N$  and  $M \otimes_2 N$  are canonically isomorphic as  $B$ -tetramodules, and*

$$M \otimes_1 N \simeq M \otimes_2 N \simeq B \otimes B^{\text{opp}} \otimes (M_0 \otimes N_0) \quad (6.11)$$

*Proof.* It is clear from (6.9) that  $M \otimes_1 N = B \otimes B^{\text{opp}} \otimes (M_0 \otimes N_0)$ , as  $B$ -tetramodule. For  $M \otimes_2 N$ , we show firstly that  $M_0 \boxtimes_2 N_0 \subset M \otimes_2 N$ , that is (6.10) is satisfied when all  $m_i \in M_0$ ,  $n_i \in N_0$ . Indeed, it follows from  $\Delta_r(m_i) = m_i \otimes 1$ ,  $\Delta_\ell(n_i) = 1 \otimes n_i$ . Then  $((B \otimes 1^{\text{opp}}) \otimes M_0) \boxtimes_2 ((1 \otimes B^{\text{opp}}) \otimes N_0)$  also satisfies (6.10). Then  $(B \otimes M_0) \boxtimes_2 (N_0 \otimes B^{\text{opp}}) \subset M \otimes_2 N$ . We need to show that no other elements belong to  $M \otimes_2 N$ . It is clear by the application of the counit map to suitable factors.  $\diamond$

Thus, when  $B$  is a Hopf algebra, we denote  $\otimes_1 = \otimes_2 = \otimes_*$ .

The Eckmann-Hilton map  $\eta_{MNPQ}: (M \otimes_2 N) \otimes_1 (P \otimes_2 Q) \rightarrow (M \otimes_1 P) \otimes_2 (N \otimes_1 Q)$  (constructed in [Sh1, Section 2.2.3]) becomes a map

$$\eta_{MNPQ}: M \otimes_* N \otimes_* P \otimes_* Q \rightarrow M \otimes_* P \otimes_* N \otimes_* Q \quad (6.12)$$

which is defined as the identities on the first and on the fourth factors, and as a map

$$\lambda_{NP}: N \otimes_* P \rightarrow P \otimes_* N \quad (6.13)$$

We compute, by Corollary 6.7:

$$N \otimes_* P = (B \otimes N_0) \otimes (P_0 \otimes B^{\text{opp}}), \quad P \otimes_* N = (B \otimes P_0) \otimes (N_0 \otimes B^{\text{opp}})$$

When we already factorized  $N$  and  $P$  as  $B \otimes N_0 \otimes B^{\text{opp}}$  and  $B \otimes P_0 \otimes B^{\text{opp}}$ , correspondingly, there is a very simple map

$$\lambda'((b \otimes n_0) \otimes (p_0 \otimes b')) = (b' \otimes p_0) \otimes (n_0 \otimes b) \quad (6.14)$$

Although this way of writing  $\lambda'$  is completely symmetric, which means that  $\lambda'_{NP} \circ \lambda'_{PN} = \text{id}$ , it is not the case when we write the same  $\lambda'$  as a map

$$\lambda: N \otimes_* P \rightarrow P \otimes_* N \quad (6.15)$$

it fails to be symmetric (but it is a braided map, that is, it satisfies the triangle axiom).

The matter is that the decomposition  $\beta: N \xrightarrow{\sim} B \otimes N_0 \otimes B^{\text{opp}}$  of Corollary 6.7 depends on the antipode.

The formula for  $\beta$  is, by Key-Lemma 6.3(ii)

$$\beta(n) = (\text{id}_B \otimes P \otimes \text{id}_{B^{\text{opp}}}) \circ (\Delta_\ell \Delta_r(n)) \quad (6.16)$$

where  $P$  is given by (6.6) (after replacing of  $B$  in (6.6) by  $B \otimes B^{\text{opp}}$ ):

$$P: N \xrightarrow{\Delta_\ell \Delta_r} B \otimes N \otimes B^{\text{opp}} \xrightarrow{S \otimes \text{id}_N \otimes S} B \otimes N \otimes B^{\text{opp}} \xrightarrow{m_\ell m_r} N \quad (6.17)$$

and, in fact,

$$P(N) = N_0 \quad (6.18)$$

Therefore, the map  $\lambda_{NP}: N \otimes_1 P \rightarrow P \otimes_1 N$  is:

$$\begin{aligned} \lambda_{NP}: N \otimes_1 P &\xrightarrow{\beta_N \otimes_1 \beta_P} (B \otimes N_0 \otimes B^{\text{opp}}) \otimes_1 (B \otimes P_0 \otimes B^{\text{opp}}) \xrightarrow{\lambda'} (B \otimes P_0 \otimes B^{\text{opp}}) \otimes_1 (B \otimes N_0 \otimes B^{\text{opp}}) \\ &\xrightarrow{(m_\ell^P m_r^P) \otimes_1 (m_\ell^N m_r^N)} P \otimes_1 N \end{aligned} \quad (6.19)$$

A routine but a direct check with definition of  $\eta_{MNPQ}$ , given in [Sh1, Section 2.2.3], proves

LEMMA 6.10. *Let  $B$  be a Hopf algebra over  $k$ , and let  $M, N, P, Q$  be four  $B$ -tetramodules. Then, after identification  $\otimes_1 = \otimes_2$ , given in Lemma ??, the Eckmann-Hilton map*

$$\eta_{MNPQ}: M \otimes_1 N \otimes_1 P \otimes_1 Q \rightarrow M \otimes_1 P \otimes_1 N \otimes_1 Q$$

is given as

$$\eta_{MNPQ} = \text{id}_M \otimes_1 \lambda_{NP} \otimes_1 \text{id}_Q$$

This map is an isomorphism of tetramodules, for any  $M, N, P, Q$ .

◇

Proposition 6.8 and Theorem 6.1 are proven.

◇

It is worth to give a general comment on the above proof. The situation with the 2-fold monoidal category  $\mathcal{Tetra}(B)$ , where  $B$  is a Hopf algebra, gives a perfect illustration for the following result, due to Joyal and Street [JS]:

THEOREM 6.11 (Joyal-Street). *Suppose  $\mathcal{C}$  be an  $n$ -fold monoidal category, for which all Eckmann-Hilton maps  $\eta_{i,j}$ ,  $1 \leq i < j \leq n$ , are isomorphisms. Then, when  $n = 2$ , the 2-fold monoidal structure comes, up to an equivalence, from a braided monoidal structure on  $\mathcal{C}$  in the following*

way. Let  $\otimes_*$  be the monoidal product in the braided monoidal category  $\mathcal{C}$ , and  $\lambda_{MN}: M \otimes_* N \rightarrow N \otimes_* M$  the braiding. Then  $\otimes_1 = \otimes_2 = \otimes_*$ , and the Eckmann-Hilton maps

$$\eta_{MNPQ}: M \otimes_* N \otimes_* P \otimes_* Q \rightarrow M \otimes_* P \otimes_* N \otimes_* Q$$

is defined as

$$\eta_{MNPQ} = \text{id}_M \otimes_* \lambda_{NP} \otimes_* \text{id}_Q$$

When  $n > 2$ , the  $n$ -fold monoidal category  $\mathcal{C}$  is a symmetric monoidal category.

It might be possible if for some associative bialgebras without antipode the compatibility (Definition 5.1) and the non-degeneracy (Definition 5.2) are also fulfilled, as they are formulated in a *homotopical* form. It seems an interesting question for further study.

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UNIVERSITEIT ANTWERPEN, CAMPUS MIDDELHEIM, WISKUNDE EN INFORMATICA, GEBOUW G  
MIDDELHEIMLAAN 1, 2020 ANTWERPEN, BELGIË

*e-mail:* Boris.Shoikhet@ua.ac.be